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A q-analogue of the supersymmetric oscillator and its q-superalgebra

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Abstract. A q-analogue of the supersymmetric oscillator is constructed out of q-boson and q-fermion creation and annihilation operators. For a fixed $n_B + n_F = 2n$, (where n_B and n_F are q-boson and q-fermion occupation numbers), the irreducible representations of the q-superalgebra generated by the q-oscillator are $(2n+1)$-dimensional. Particular cases when q is a root of ± 1 are discussed. A realization of the quantum group $SU_q(2)$ is obtained using a pair of 1-fermion creation and annihilation operators.

Quantum deformations of Lie algebras and Lie groups [1] have been developed in the theory of integrable systems where the Yang-Baxter equation plays a crucial role [2]. Quantum groups have found applications in lattice statistical models at the critical temperature [3] and in 2D conformal field theories [4].

One of the simplest examples of a quantum group is $SU_q(2)$, the q-deformation of SU(2). A realization of $SU_q(2)$, using a q-analogue of the bosonic harmonic oscillator and the Jordan–Schwinger mapping has been achieved by Macfarlane [5] and Biedenharn [6]. Using the same realization a theory of tensor operators for $SU_q(2)$ is constructed in [7].

In this paper we consider a q-analogue of the supersymmetric oscillator. Naturally, in addition to q-boson creation and annihilation operators, we introduce q-fermion equivalents, whose anticommutation relation involves q-extension. In the ordinary supersymmetric theory the superalgebra [8] is generated by H, Q_+ and Q_-, where H (Hamiltonian) is the even generator and $Q_\pm$ are the odd generators of the superalgebra. These satisfy the commutation relations,

$$\{Q_+, Q_-\} = H \qquad [Q_\pm, H] = 0. \tag{1}$$

The q-deformation of this algebra turns out to be quite interesting as we shall show below. For arbitrary values of q, any number of q-fermions can occupy the same state. We are thus able to construct a q-realization of $SU_q(2)$ using a pair of q-fermion creation and annihilation operators. We hope that this construction will provide us with a method to construct a q-deformation of other superalgebras.

The q-creation and annihilation operators of the bosonic oscillator are postulated to satisfy the commutation relations

$$a_q a_q^\dagger - \sqrt{q}\, a_q^\dagger a_q = q^{-N_B/2} \tag{2}$$

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where $q \in \mathbb{C}$, and N_B is the boson number operator satisfying

$$[N_B, a_q^\dagger] = a_q^\dagger \qquad [N_B, a_q] = -a_q.$$

It should be noted that, because of (2), $N_B \neq a_q^\dagger a_q$. Only in the limit $q=1$, $N_B = a^\dagger a$. Iterating relation (2) we find,

$$a_q(a_q^\dagger)^n - q^{n/2}(a_q^\dagger)^n a_q = \sum_{k=0}^{n-1} q^{k/2}(a_q^\dagger)^k q^{-N_B/2}(a_q^\dagger)^{n-k-1}. \tag{3}$$

This relation can be used to construct an orthonormal n q-boson state:

$$|n\rangle_q^B = ([n]_q^B!)^{-1/2}(a_q^\dagger)^n|0\rangle^B \tag{4}$$

where

$$[n]_q^B! = [n]_q^B[n-1]_q^B \ldots [1]_q^B \tag{5}$$

and

$$[n]_q^B = q^{-(n-1)/2}\sum_{k=0}^{n-1} q^k = \frac{q^{1/2}}{(1-q)}(q^{-n/2} - q^{n/2}). \tag{6}$$

The vacuum state $|0\rangle^B$ has the property

$$a_q|0\rangle^B = 0. \tag{7}$$

The following relations can be verified.

$$\begin{aligned} N_B|n\rangle_q^B &= n|n\rangle_q^B \\ a_q^\dagger|n\rangle_q^B &= \sqrt{[n+1]_q^B}\,|n+1\rangle_q^B \\ a_q|n\rangle_q^B &= \sqrt{[n]_q^B}\,|n-1\rangle_q^B. \end{aligned} \tag{8}$$

We now introduce q-fermion creation and annihilation operators $f_q^\dagger$ and f_q respectively, postulated to satisfy the q-anticommutation relation

$$f_q f_q^\dagger + q^{1/2} f_q^\dagger f_q = q^{-N_f/2} \tag{9}$$

where N_f is the q-fermion number operator satisfying the relations

$$[N_f, f_q^\dagger] = f_q^\dagger \qquad [N_f, f_q] = -f_q. \tag{10}$$

Iterating (9) one obtains

$$f_q(f_q^\dagger)^n + (-1)^{n-1}q^{n/2}(f_q^\dagger)^n f_q = \sum_{k=0}^{n-1}(-1)^k q^{k/2}(f_q^\dagger)^k q^{-N_F/2}(f_q^\dagger)^{n-k-1}. \tag{11}$$

The orthonormal n q-fermion state can be defined by

$$|n\rangle_q^F = \frac{1}{([n]_q^F!)^{1/2}}(f_q^\dagger)^n|0\rangle \tag{12}$$

where

$$[n]_q^F! = [n]_q^F[n-1]_q^F \ldots [1]_q^F \tag{13}$$

and

$$[n]_q^F = q^{-(n-1)/2}\sum_{k=0}^{n-1}(-q)^k = \frac{q^{1/2}}{1+q}(q^{-n/2} - (-1)^n q^{n/2}). \tag{14}$$

For $q=1$, it is seen from (12) and (13) that only $|0\rangle$ and $f^{\dagger}|0\rangle$ are non-vanishing while $(f^{\dagger})^n|0\rangle$ $(n>1)$ vanish identically. We interpret this as $(f^{\dagger})^2=0$ $(q=1)$. It is also interesting to note from (14) that when $q=\exp(\pm 2\pi i/n)$ for n even, $[n]_q^{F}=0$. Similarly when $q=\exp(\pm i\pi/n)$ (n odd) $[n]_q^{F}=0$. For these q values, the q-fermion states are truncated at the nth level. For example, when $q=e^{\pm i\pi/3}$, no more than two q-fermions can occupy a given state. Relations analogous to (8) can be obtained for the q-fermion number operator N_F by simply replacing $[n]_q^{B}$ by $[n]_q^{F}$. Note that, once again, $N_F \neq f_q^{\dagger} f_q$.

If the Hamiltonian of the q-fermion oscillator is taken as

$$H_q^{F}=\tfrac{1}{2}\hbar\omega(f_q^{\dagger}f_q-f_q f_q^{\dagger}) \tag{15}$$

then

$$H_q^{F}|n\rangle_q^{F}=\tfrac{1}{2}\hbar\omega[[n]_q^{F}-[n+1]_q^{F}]|n_q^{F}. \tag{16}$$

As in the q-boson oscillator case, the levels are not equally spaced.

Consider now the q-generalized supersymmetric oscillator. Construct the normalized basis states

$$|n_B, n_F\rangle=([n^B]_q^B![n^F]_q^F!)^{-1/2}(a_q^{\dagger})^{n_B}(f_q^{\dagger})^{n_F}|0\rangle. \tag{17}$$

We introduce the odd generators $Q_{\pm}$ by

$$Q_+=a_q f_q^{\dagger} \qquad Q_-=a_q^{\dagger} f_q. \tag{18}$$

We assume that a_q and f_q commute with each other. From (17) we have

$$\begin{aligned} Q_+|n_B, n_F\rangle &= ([n_B]_q^B[n_F+1]_q^F)^{1/2}|n_B-1, n_F+1\rangle \\ Q_-|n_B, n_F\rangle &= ([n_B+1]_q^B[n_F]_q^F)^{1/2}|n_B+1, n_F-1\rangle \end{aligned} \tag{19}$$

Q_+ and Q_- convert a q-boson into a q-fermion and vice versa.

We introduce the even generators N and S by

$$N=\tfrac{1}{2}(N_F+N_B) \qquad S=\tfrac{1}{2}(N_F-N_B). \tag{20}$$

The following relations provide a realization of a q-superalgebra.

$$\begin{aligned} &[N, Q_{\pm}]|n_B, n_F\rangle=0 \\ &[S, Q_{\pm}]|n_B, n_F\rangle=\pm Q_{\pm}|n_B, n_F\rangle \\ &\{Q_+, Q_-\}|n_B, n_F\rangle \\ &\quad \equiv H|n_B, n_F\rangle \\ &\quad =(q^{-n_B/2}[n^F]_q^F+q^{-n_F/2}[n^B]_q^B)|n_B, n_F\rangle \\ &\quad =\left\{\frac{1}{(q^{1/2}+q^{-1/2})}[q^{-N}-(-1)^{N+S}q^S]+\frac{1}{q^{1/2}-q^{-1/2}}(q^{-S}-q^{-N})\right\}|n_B n_F\rangle \\ &[N, S]|n_B, n_F\rangle=0. \end{aligned} \tag{21}$$

These relations illustrate the general structure of a superalgebra, $[E, E]\sim E$, $[E, 0]\sim 0$, $\{0, 0\}\sim E$ for the odd 0 and even E generators. The abstract commutation and anticommutation relations of the q-deformed superalgebra can be read off from (21). It is seen that this algebra is generated by the set $\{N, S, Q_+, Q_-\}$. N and S generate two commuting U(1) groups, while the odd generators contain both N and S in their anticommutator. The U(1) group generated by S decouples from the superalgebra and the Hamiltonian coincides with the number operator for the case $q=1$. Since $Q_\pm$ commute with only N, we may take the 'Hamiltonian' of the q-super oscillator to be the total number operator N, so that one has q-supersymmetry for the system. If the Hamiltonian H is defined as $\{Q_+, Q_-\}$ in analogy with the $q=1$ case, then H is given in (21) by $H|n_b, n_F\rangle = (q^{-N_B/2}[n_F]_q^F + q^{-N_F/2}[n_B]_q^B)|n_B, n_F\rangle$. This is an interesting generalization of the $q=1$ case where $H = N_B + N_F$. However H does not commute with $Q_\pm$ and thus is not invariant under the q-superalgebra. One finds a similar property for the two-dimensional q-harmonic oscillator studied in [5] and [6]. The $SU_q(2)$ generators do not commute with the Hamiltonian of the oscillator. Nevertheless, since the deformation of SU(2) has the property that it preserves the dimensions of highest weight representations, the 2D harmonic oscillator energy eigenstates can be classified by $SU_q(2)$ highest weight representations. The associativity of the q-superalgebra in (21) is easily verified by checking the generalized Jacobi identities.

It is useful to rewrite the basis states as follows. Let n and s denote the eigenvalues of N and S respectively. Then we can write $|n_B, n_F\rangle = |n, s\rangle$, where $n_B = n-s$, and $n_F = n+s$. Then

$$\begin{aligned} Q_+|n, s\rangle &= \sqrt{[n-s]_q^B[n+s+1]_q^F}\,|n, s+1\rangle \\ Q_-|n, s\rangle &= \sqrt{[n-s+1]_q^B[n+s]_q^F}\,|n, s-1\rangle. \end{aligned} \tag{22}$$

$Q_\pm$ change the s value of the state $|n, s\rangle$ by one. For a given n, it is clear from the definition of the operators N and S that $-n \leqslant s \leqslant n$. When $s=-n$, the state $|n, -n\rangle$ (the highest weight state) is completely bosonic, while for $s=n$, the state $|n, n\rangle$ is totally fermionic. Thus $Q_\pm$ are nilpotent:

$$(Q_\pm)^{2n+1} = 0. \tag{23}$$

Thus for a give n, the irreducible representations of the q-superalgebra are $(2n+1)$ dimensional. This is of course true when $q \neq 1$ or one of the roots of (± 1) as indicated earlier. If $q=1$, $Q_\pm^2 = 0$, and all representations are two dimensional. When $q = e^{\pm i\pi/m}$ (m odd), the irreducible representations are $(2m+1)$ dimensional of $m<n$ and are $(2n+1)$ dimensional of $m \geqslant n$. The same is true when $q = e^{\pm 2\pi i/m}$ (m even).

Finally, we construct a realization of $SU_q(2)$ using a pair of q-fermion creation and annihilation operators. This is analogous to the construction in [5] using bosonic operators. Let $(f_{1q} f_{2q}, f_{1q}^\dagger f_{2q}^\dagger)$ be the annihilation and creation operators for a pair of mutually anticommuting q-fermions. Form the quantities

$$\begin{aligned} K_+ &= \exp[-i\pi(N_1+N_2-1)/4] f_i^\dagger(\sigma^+)_{ij} f_j \\ K_- &= \exp[-i\pi(N_1+N_2-1)/4] f_i^\dagger(\sigma^-)_{ij} f_j \qquad i, j = 1, 2 \\ K_3 &= \tfrac{1}{2}(N_1 - N_2) \end{aligned} \tag{24}$$

where $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Let us take the basis states as

$$|k, n\rangle_q^F = |k+n, k-n\rangle_q^F = ([k+n]_q^F![k-n]_q^F!)^{-1/2} (f_{1q}^\dagger)^{k+n} (f_{2q}^\dagger)^{k-n} |0\rangle_q \tag{25}$$

which have the property that $K_3|k, n\rangle_q^F = n|k, n\rangle_q^F$ and $K_\pm|k, n\rangle_q^F \sim |k, n\pm 1_q^F$. It can be verified that

$$\begin{aligned} [K_+, K_-]|k, n\rangle_q^F &= \exp(i\pi/2)\frac{[(-q)^{-K_3} - (-q)^{K_3}]}{(q^{1/2}+q^{-1/2})}|k, n\rangle_q^F \\ [K_3, K_\pm]|k, n\rangle_q^F &= \pm K_\pm|k, n\rangle_q^F \end{aligned} \tag{26}$$

As in the case of $SU_q(2)$ generators constructed in [8] out of q-boson creation-annihilation operators, these commutation relations are verified only on kets that terminate with the q-vacuum state. For $q=1$, it is readily seen that (26) verifies the fundamental ($n=\pm\frac{1}{2}$) representation of SU(2). (This is the only representation of SU(2) that can be constructed out of f_1 and f_2.) For $q \neq 1$, (26) can be brought to the standard form

$$[K_+, K_-] = \frac{(q')^{K_3} - (q')^{-K_3}}{(q')^{1/2} - (q')^{-1/2}} \tag{27}$$

$$[K_3, K_\pm] = \pm K_\pm \tag{28}$$

by replacing q by $e^{i\pi}(q')^{-1}$.

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Note added in proof. In general, equation (12) must read $|n\rangle_q^F = (|[n_q^F]!|)^{-1/2}(f_q^\dagger)^n|0\rangle$, so as to be applicable for all values of q. This is because for q outside the range (0, 1), $[n_q^F]$ may become negative or complex for certain values of n. This prescription for defining the norm amounts to defining the dual vector to $(f^\dagger)^n|0\rangle$ to be $e^{i\theta(n)}\langle 0|f^n$ where $\theta(n)$ is the argument of $[n_q^F]!$.

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